Let $\rho \equiv r/R$ so that the cubic becomes

$$\rho^3 - \rho^2 + 1/(3n) = 0. \tag{3}$$

Since the constant term of this equation is positive there is always one real negative root, and this is inadmissible. Therefore the remaining roots must be real in order that a positive root may exist. The derivative of the left-hand member of (3) is $\rho(3\rho - 2)$ which vanishes for $\rho = 0$ and $\rho = 2/3$. The expression for the derivative shows that there is a maximum for $\rho = 0$, and a minimum for $\rho = 2/3$; the value of the minimum is $1/3(1/n - 4/9)$ and hence if $1/n > 4/9$ there is only one real root, the negative root already mentioned.

Consequently the given value $1/n = 3/4$ conflicts with the preceding condition and makes the problem impossible.

Also solved by T. M. Blakslee, L. A. Eastburn, R. A. Johnson, R. H. Marshall, J. Q. McNatt, Arthur Pelletier, and A. V. Richardson.

2855 [1920, 377]. Proposed by J. L. Riley, Stephenville, Texas.

Show that the circle of curvature at any point of the ellipse cannot pass through the centre unless the eccentricity be greater than $1/\sqrt{2}$.

I. Solution by A. V. Richardson, Bishop's College, Lennoxville, Quebec.

In the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b),$$

the normal at the point $(a \cos \theta, b \sin \theta)$ is

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2. \tag{1}$$

To find the coördinates of the center of curvature, we solve (1) and its derivative with respect to θ,

$$\frac{ax \sin \theta}{\cos^2 \theta} + \frac{by \cos \theta}{\sin^2 \theta} = 0,$$

giving

$$x = \frac{a^2 - b^2}{a} \cos^3 \theta; \qquad y = -\frac{a^2 - b^2}{b} \sin^3 \theta. \tag{2}$$

The (radius of curvature)^2 is, of course, the (distance)^2 between this point and $(a \cos \theta, b \sin \theta)$ *i.e.*,

$$\frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^3}{a^2 b^2}.$$

If we put the radius of curvature equal to the distance of the center of curvature from the origin we shall get after some reductions

$$(a^2 - 2b^2) \cos^2\theta = (2a^2 - b^2) \sin^2\theta. \tag{3}$$

Hence, for real values of θ, we must have $a^2 > 2b^2$, *i.e.*,

$$a^2 > 2a^2 (1 - e^2) \qquad \text{or} \qquad e > \frac{1}{\sqrt{2}}.$$

II. Remarks by Otto Dunkel, Washington University.

If we determine the condition that the perpendicular bisector of the segment from the origin to the point $(a \cos \theta, b \sin \theta)$ shall pass through the center of curvature we shall get equation (3) more easily.

Or we may express this by saying that the center of curvature projects into the middle point of the segment. This condition takes the form $2(x_1 x_c + y_1 y_c) - (x_1^2 + y_1^2) = 0$, which reduces to the same equation (3).

We find also that e may equal $1/\sqrt{2}$. In the ellipse for which $e = 1/\sqrt{2}$ the radius of curvature at the extremity of the major axis is equal to $a/2$.

Also solved by A. M. Harding, William Hoover, and H. S. Uhler.